Stability and synchronism of certain coupled Dynamical Systems

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Abstract

We obtain sufficient conditions for the stability of the synchronized solution for certain classes of coupled dynamical systems. This is accomplished by finding an analytic expression for the transverse Lyapunov exponent through spectral analysis. We then indicate some applications to population dynamics.

1 Introduction

The study of coupled dynamical systems has received considerable attention recently for its interest from the mathematical, physical and biological point of view, see for instance [19], [2], [4], [25], [22] and [6] among others. One concern about such systems is to whether or not they will present synchronization phenomena and to whether or not such synchronization is stable.

The system below describes the evolution of a system consisting of d identical subsystems where, on every iteration, each subsystem undergoes its common local evolution determined by f, followed by a density dependent coupling process encoded by C and φ. The system can model a population consisting of d patches, x_j, $j = 1, \ldots, d$, where, in the absence of migration, patch j is controlled by a *local dynamics* $x^j_{t+1} = f(x^j_t)$. When migration is present, $\varphi(f(x^j_t))$ individuals leave patch j and are distributed with density c_{ji} on patch i. The *global dynamics* is then

$$x^j_{t+1} = f(x^j_t) - \varphi(f(x^j_t)) + \sum_{i=1}^{d} c_{ji}\varphi(f(x^i_t));\;\; i,j = 1,\cdots,d. \qquad (1)$$

Here f is a C^1 map on $[0,\infty)$, $C=[c_{ij}]$ is doubly stochastic, that is, $c_{ij}\geq 0$ and $\forall i,j, \displaystyle\sum_{i=1}^{d} c_{ij} = \sum_{j=1}^{d} c_{ij} = 1$. Furthermore we will assume φ differentiable a. e. with φ' bounded. The models in population dynamics considered in [2], [4], [5], [23], [11], [22] are all particular cases of (1) for special choices of C and φ.

The condition on C being doubly stochastic reflects that there are no losses during the migration process. It is also necessary for the invariance of the diagonal of the phase space, that is, for $x_t^j = x_t^i = x_t$ to be a solution of (1), where each x_t^j satisfies $x_{t+1}^j = f(x_t^j)$.

In this paper we obtain sufficient conditions for the stability of the aforementioned synchronized solutions. The criteria involve the Lyapunov exponents of the one dimensional map f and of the codimension one transverse dynamical system to be defined below.

The paper is organized as follows. In the next section we provide a criterion for stability for general systems. In §3 we improve our result for the case of normal operators. In §4, we consider a system where the coupling/migration process is time dependent. We formulate and prove the corresponding results of the previous sections in this setting. In the last section we indicate some applications to population dynamics.

Previous results treat only the case where φ' is a constant, or a 2-valued step function as well as particular examples of matrices C and in these cases we recover such results. Our treatment only requires φ' to be bounded and C to be doubly stochastic and irreducible. This last condition is easily seen to be necessary, see below. Moreover we are unaware of any treatment of the systems in §4 in the literature. Thus we extend some of the results of [4], [5], [23], [13],[6], [11] and others to these more general situations.

2 Stability: General case

In order to understand the behavior of orbits starting at nearby points of the diagonal of the phase space, we first linearize (1). If $J_t = [\alpha_{ij}]$ denotes the Jacobian matrix of (1) restricted to the synchronized orbit, we have

$$\alpha_{ij} = \begin{cases} f'(x_t)\left(1-(1-c_{ii})\varphi'(f(x_t))\right), & \text{for } i=j \\ f'(x_t)\varphi'(f(x_t))c_{ij}, & \text{for } i\neq j \end{cases}$$

We have $J_t = f'(x_t)H_t$, where $H_t = I - \varphi'(f(x_t))B$, where $B = I - C$.

We will assume that C is irreducible. This is almost a necessary condition for otherwise it permits the existence of uncoupled unsynchronized

subsystems that are each synchronized. In this case we can apply Fröbenius theorem [10] to show that $\lambda = 1$ is the simple dominant eigenvalue of C, associated to the eigenvector $v = (1, \ldots, 1)$. This furnishes the decomposition $\mathbb{R}^d = \mathbb{R}v \oplus W$, where W is $C-$invariant $d-1$ dimensional subspace. Under these conditions

$$B = P^{-1} \begin{bmatrix} 0 & \\ & A \end{bmatrix} P \tag{2}$$

where P is the matrix of the appropriate change of basis. This decomposition implies that the stability of the synchronized solution of (1) is a consequence of the stability of the trivial solution of the transversal component, w_t, which satisfies

$$w_{t+1} = f'(x_t) \left(I - \varphi'(f(x_t))A \right) w_t. \tag{3}$$

We will show that under a certain integrability condition the map above is in fact a contraction which in turn implies the stability of $w_t \equiv 0$. The analysis of (3) will be based on the Lyapunov exponents (see [15], [16]) of (3). Define

$$K_n(x) = \prod_{k=0}^{n-1} f'(f^k(x)) \left(I - \varphi'(f^{k+1}(x))A \right)$$

where $f^0(x) = x$ and $f^k(x) = f(f^{k-1}(x))$ for $k > 0$. Clearly if $\mathcal{K} = \limsup ||K_n||^{1/n}$ satisfies $\mathcal{K} < 1$ we have that (3) is a contraction. Now observe

$$K_n(x) = L_n(x)\Lambda_n(x) = \left(\prod_{k=0}^{n-1} f'(f^k(x)) \right) \left(\prod_{k=0}^{n-1} I - \varphi'(f^{k+1}(x))A \right) \tag{4}$$

$L_n(x)$ depends only on the local dynamics f while $\Lambda_n(x)$ reflects also the effects of φ and C. Let ρ be an invariant measure of the local system. Define for $x > 0$, $\ln^+(x) = \max(\ln(x), 0)$. By Birkhoff's ergodic theorem, if $\ln^+|f'| \in L^1(\rho)$, there exists $\lim_n \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(f^k(x))|$ for ρ-a.e. x. For ρ ergodic, this limit, call it L, is independent of x and is given by $\int_0^\infty \ln |f'(s)| \, d\rho(s)$. L is the Lyapunov exponent of the local system governed by f.

Similarly the ergodic theorem of Oseledec [15] implies that if

$$\int_0^\infty \ln^+ ||I - \varphi'(s)A|| \, d\rho(s) < \infty,$$

there exists $\lim_n \frac{1}{n} \ln ||\Lambda_n(x)|| =: \ln \Lambda(x)$ for ρ-a.e. x and this limit is independent of x provided ρ is ergodic. Our first result is

Theorem 1 *Consider the system (1) where f is a C^1 map, φ' bounded, C doubly stochastic, irreducible such that $\ln^+ ||I-\varphi'(s)A||$ and $\ln^+ |f'(s)|$ are in $L^1(\rho)$, where ρ is an $f-$invariant measure. Let $L = \sup_x \lim_n |L_n(x)|^{1/n}$ and $\Lambda = \sup_x \lim_n ||\Lambda_n(x)||^{1/n}$. If $L\ \Lambda < 1$, there exists a set E with $\rho(E) = 1$ such that for all $x \in E$, the synchronized solution of (1) is asymptotically stable.*

Proof: By Oseledec's theorem there exists a set E with $\rho(E) = 1$ such that for all x in E, $\lim_n ||\Lambda_n(x)||^{1/n} = \Lambda(x)$. We claim that for all x in E

$$\Lambda(x) \leq \exp\left(\int_0^\infty \ln^+ ||I - \varphi'(s)A||\ d\rho(s)\right) \tag{5}$$

By the continuity of the norm and the function $\ln(\cdot)$, the dominated convergence theorem implies that we only need to prove (5) for φ' simple.

Let $\varphi'(x) = \sum_{k=1}^r a_k \chi_{E_k}(x)$, where E_k are measurable and disjoint with $\rho(E_k) > 0$ and such $E \subset \bigcup_k E_k$. For $1 \leq k \leq r$, let

$$\rho_{k,n} = \frac{\sharp\{0 \leq j < n : f^j(x) \in E_k\}}{n}.$$

This gives

$$\begin{aligned}
\Lambda_n(x) &= \prod_k (I - a_k A)^{n\rho_{k,n}}, \text{ which imply} \\
||\Lambda_n(x)|| &\leq \prod_k ||I - a_k A||^{n\rho_{k,n}}, \text{ and} \\
\ln ||\Lambda_n(x)|| &\leq \sum_k n\ \rho_{k,n}\ \ln ||I - a_k A||
\end{aligned}$$

Birkhoff's ergodic theorem applied to χ_{E_k} shows that for $1 \leq k \leq r$, we have

$$\lim_n \rho_{k,n} = \rho(E_k).$$

Therefore, given $\epsilon > 0$, there exists n_0 such that for $n > n_0$, we have $\rho_{k,n} \leq (1+\epsilon)\rho(E_k)$. Since $\ln(\cdot) \leq \ln^+(\cdot)$,

$$\frac{1}{n} \ln ||\Lambda_n(x)|| \leq (1+\epsilon) \sum_{k=1}^r \ln^+ ||I - a_k A||\ \rho(E_k) = (1+\epsilon) \int_0^\infty \ln^+ ||I - \varphi'(s)A||\ d\rho(s)$$

Note that the right hand side is independent of x. Thus we have

$$||K_n(x)||^{1/n} \leq |L_n(x)|^{1/n}\ ||\Lambda_n(x)||^{1/n} \leq (1+\epsilon) L \Lambda^{1+\epsilon}$$

and since $\epsilon > 0$ is arbitrary we have

$$\lim_n ||K_n(x)||^{1/n} \leq L\,\Lambda < 1$$

and thus the transversal map (3) is a contraction. $\square$

2.1 Remarks:

(i) if $E = [0, \infty)$, $L\,\Lambda \leq 1$ is necessary for the stability of the synchronized solution.

(ii) in certain cases, which include the case A semi simple, the norm $||I - \varphi'(f^k(x))A||$ is simply the spectral radius, $\sigma_{-1}(H_t)$, of the restriction of B to the subspace W. Therefore, with the same hypothesis on f, φ and C we have

$$\Lambda \leq \Lambda_1 = \int_0^\infty \sigma_{-1}(H_{\varphi'(s)})\, d\rho(s)$$

and thus $L\Lambda_1 < 1$ is a sufficient condition for the stability of synchronized solution of (1).

3 Stability: normal operators

In the case of normal operators, that is, $AA^* = A^*A$, one can improve the previous result with the help of the functional calculus of [14] and [20]. The proof of Oseledec's theorem in [21] shows that $\lim_n (\Lambda_n^* \Lambda_n(x))^{\frac{1}{2n}} = \Lambda(x)$ in operator norm. Spectral analysis of $\Lambda(x)$ determines the Lyapunov exponents and respective subspaces. Even though in our case C represents a $d \times d$ matrix, the discussion below is valid for any bounded normal operator in a Hilbert space. We present the results in this generality since it will be applied in the future to systems more general than (1). We will show that for A normal, we have

$$\Lambda(x) = \exp\left(\int_0^\infty \ln|I - \varphi'(s)A|\, d\rho(s)\right) \tag{6}$$

Without loss of generality we can assume that for s in E, $I - \varphi'(s)A$ is nonsingular. This implies that for all $\lambda \in \sigma(A)$, $\ln|1 - \lambda\varphi'(s)|$ is continuous on E. The spectral theorem for bounded normal operators (Theorem 12.23

in [20]), allow us to define $|I - \varphi'(s)A|$ and $\ln|I - \lambda\varphi'(s)A|$ by

$$\begin{aligned} A &= \int_{\sigma(A)} \lambda \, dP(\lambda) \\ \ln|I - \varphi'(s)A| &= \int_{\sigma(A)} \ln|1 - \lambda\varphi'(s)| \, dP(\lambda) \end{aligned}$$

where $dP(\lambda)$ are the spectral projections associated with A. Our hypothesis on the spectrum of $I - \varphi'(s)A$ together with Fubini's theorem imply

$$\int_0^\infty \ln|I - \varphi'(s)A| \, d\rho(s) = \int_{\sigma(A)} \left(\int_0^\infty \ln|1 - \lambda\varphi'(s)| \, d\rho(s) \right) \, dP(\lambda)$$

and since this last integral defines a bounded operator, we take exponentials to obtain

$$\exp\left(\int_0^\infty \ln|I - \varphi'(s)A| \, d\rho(s) \right) = \exp\left(\int_{\sigma(A)} \left(\int_0^\infty \ln|1 - \lambda\varphi'(s)| \, d\rho(s) \right) \, dP(\lambda) \right)$$

and thus the right hand side of (6) is well defined. The spectral mapping theorem then implies

$$\sigma\left(\exp\left(\int_0^\infty \ln|I - \varphi'(s)A| \, d\rho(s) \right) \right) = \left\{ \exp\left(\int_0^\infty \ln|1 - \lambda\varphi'(s)| \, d\rho(s) \right) : \lambda \in \sigma(A) \right\}$$

Our next result is

Theorem 2 *Let f, φ', C, and L be as in Theorem 1. Let Λ be the spectral radius of (6). Then, if $L\ \Lambda < 1$, the synchronized solution of (1) is asymptotically stable.*

Proof: The above considerations imply that in order to prove the theorem we only need to establish (6). We will do so first assuming φ' simple. Then a standard limit process extends the result to φ' in L^∞. Let $\varphi' = \sum_{k=1}^N \varphi_k \chi_{E_k}$ where E_k are disjoint measurable. Note that since φ' is bounded, the integral in (6) is well defined. As before we define $\rho_{k,n} = \dfrac{\sharp\{0 \le j < n : f^j(x) \in E_k\}}{n}$. Then

$$(\Lambda_n^* \Lambda_n)^{\frac{1}{2n}} = \prod_{k=1}^N |I - \varphi_k A|^{\rho_{k,n}} = \exp\left(\sum_{k=1}^N \rho_{k,n} \ln|I - \varphi_k A| \right)$$

Since for all k, $\lim_n \rho_{n,k} = \rho(E_k)$, the continuity of exp(.) and the Lebesgue dominated convergence theorem imply

$$\lim_n \exp\left(\sum_{k=1}^{N} \rho_{k,n} \ln|I - \varphi_k A|\right) = \exp\left(\int_0^\infty \ln|I - \varphi'(s)A|\, d\rho(s)\right).$$

proving (6). This finishes the proof of Theorem 2.□

3.1 Remarks:

(i) Since for all $\lambda \in \sigma(A)$, $|1 - \lambda\varphi'(s)| \leq ||I - \varphi'(s)A||$, we have $\Lambda \leq \exp\left(\int_0^\infty \ln^+ ||I - \varphi'(s)A||\, d\rho(s)\right)$, therefore Theorem 2 is an improvement of Theorem 1.

(ii) Note that all the effects of the weighted network relevant to synchronization are reflected on the spectrum of Λ and can therefore, in certain cases, be independent of d. It would be interesting to extend such results to the $d = \infty$ case since Fröbenius theorem as well as Oseledec's ergodic theorem are not valid without further assumptions.

4 Extensions

The analysis above can be applied to more general systems. One such instance is the case below where the coupling/migration process no longer depends on the density but instead obeys a seasonal dynamics, that is, on each cycle a time dependent fraction μ_t of each patch will mix according to a time dependent distribution $C_t = [c_{ij}^t]$ as follows

$$x_{t+1}^j = f(x_t^j) - \mu_t\, f(x_t^j) + \sum_{i=1}^{d} c_{ji}^t\, \mu_t\, f(x_t^i);\;\; i,j = 1, \cdots, d. \tag{7}$$

Again f is a C^1 map on $[0,\infty)$. For g continuous on $[0,1]$ and μ_0 arbitrary, we assume that $\mu_t = \mu_t(\mu_0)$ is given by $\mu_{t+1} = g(\mu_t)$. Similarly let $\{C_s\}_{s\in[0,1]}$ be a family of doubly stochastic irreducible matrices. If h is a continuous map on $[0,1]$ and s_0 is arbitrary, define $s_{t+1} = h(s_t)$ and $C_t = C_t(s_0)$ by $C_{t+1} = C_{h(s_t)}$. Assume that $G(\mu, s) = \mu\, C_s$ is a measurable operator valued map with respect to the product measure $\nu \times \eta$ on $[0,1] \times [0,1]$, where ν and η are, respectively, a g-invariant and h-invariant ergodic measures on $[0,1]$. Under these assumptions the diagonal of the phase space, $x_t^i =$

$x_{t+1}^{j} = x_t$, is a synchronized solution and we are interested in its stability. This system falls under the general theory of Random Dynamical Systems as presented in [1] and thus obey, under the appropriate integrability condition, a mutiplicative ergodic theorem. Our approach is direct and avoid the use of this theory. Moreover, due to the particular nature of (7), we are able, as in the previous sections, to obtain much more precise information about the limit operators than is given by the ergodic theorem alone.

The Jacobian matrix of (7), $J_t = [\alpha_{ij}^t]$ is now given by

$$\alpha_{ij} = \begin{cases} f'(x_t)\left(1 - \mu_t(1 - c_{ii}^t)\right), & \text{for } i = j \\ f'(x_t)\, \mu_t\, c_{ij}^t, & \text{for } i \neq j \end{cases}$$

yielding $J_t = f'(x_t)(I - \mu_t B_t)$ where $B_t = I - C_t$. Our first task is to decompose the linearized system into diagonal and transversal components and then estimate the respective Lyapunov exponents by the appropriate ergodic theorem. This cannot be done in general for arbitrary families $\{C_t\}$, therefore we will consider special cases of increasing generality.

4.1 Simultaneous Diagonalizable Matrices

This include the families of commuting symmetric matrices and of circulant matrices. In this case there exists a matrix P such that for all t, $B_t = P^{-1}\, M_t\, P$ where M_t is a diagonal matrix with entries $\{1, \lambda_2(t), \ldots, \lambda_d(t)\}$ in its diagonal. Each $\lambda_j(t)$ is measurable and satisfies $|\lambda_j(t)| \leq 1$ and $\lambda_j(t) \neq 1$ by Fröbenius theorem. In the symmetric (commuting) case the $\lambda_j(t)$ are real and lie in $(-1, 1)$. We obtain that the transversal component of (7) satisfies

$$w_{t+1} = f'(x_t)\,(I - \mu_t\, D_t)\, w_t. \tag{8}$$

where D_t is the $(d-1)$ dimensional $\{\lambda_2(t), \ldots, \lambda_d(t)\}$ diagonal matrix. As before, the synchronized solution of (7) is stable if and only if $w_t = 0$ is a stable solution of (8). For each x, μ_0, s_0 in $[0, 1]$, define $L_n(x)$ as in (4), and for $j = 2, \ldots, d$, define

$$\Lambda_{j\,n}(\mu_0,\, s_0) = \prod_{k=0}^{n-1} |1 - g^k(\mu_0)\, \lambda_j(h^k(s_0))|$$

The analogue of Theorem 2 is

Theorem 3 *Consider the system (7) under the above conditions. Assume in addition*

(i) $\ln^+ |f'(s)|$ *belongs to* $L^1(\rho)$*, where* ρ *is a* $f-$*invariant measure on* $[0,\infty)$.

(ii) *For all* $2 \leq j \leq d$, $\ln^+ |1 - \mu\, \lambda_j(s)|$ *belong to* $L^1(\nu \times \eta)$.

Then there exist sets $E \subset [0,\infty)$ *and* $F \subset [0,1] \times [0,1]$ *with* $\rho(E) = (\nu \times \eta)\,(F) = 1$*, such that for all* $x \in E$ *and* $(\mu_0, s_0) \in F$*, the limits* $L(x) = \lim_n |L_n(x)|^{1/n}$ *and* $\Lambda_j(\mu_0, s_0) = \lim_n |\Lambda_{j\,n}(\mu_0, s_0)|^{1/n}$ *exist. Moreover if* $L = \sup_x L(x)$ *and* $\Lambda = \sup_{j,\mu_0,s_0} \Lambda_j(\mu_0, s_0)$ *satisfy* $L\ \Lambda < 1$*, the synchronized solution of (7) is asymptotically stable.*

Proof: The existence of $L(x)$ for x in a set of full measure follows, as before, from Birkhoff's ergodic theorem applied to $\ln L_n(\cdot)$ and (i). In addition the limit is independent of x provided ρ is ergodic. The decomposition (8) allow us to derive the transversal component directly without making use of Oseledec's theorem. For $2 \leq j \leq d$, $(\mu_0, s_0) \in [0,1] \times [0,1]$, write

$$\Lambda_{j\,n}(\mu_0, s_0) = \exp\left(\frac{1}{n}\sum_{k=0}^{n-1} \ln |1 - g^k(\mu_0)\, \lambda_j(h^k(s_0))|\right)$$

Condition (ii) and a $(d-1)$-fold application of Birkhoff's ergodic theorem imply the existence of a set F, with $\nu \times \eta\,(F) = 1$ such that for all j, all $(\mu_0, s_0) \in F$, there exists $\Lambda_j(\mu_0, s_0) = \lim_n \Lambda_{j\,n}(\mu_0, s_0)$. If, in addition, $\nu \times \eta$ is ergodic, the limit is independent of (μ_0, s_0) and is given by

$$\Lambda_j = \exp\left(\int_{[0,1]\times[0,1]} \ln |1 - \mu\, \lambda_j(s)|\ d(\nu \times \eta)(\mu, s)\right) \tag{9}$$

This gives $diag\{\Lambda_2, \ldots, \Lambda_d\}$ as the analogue of (6). Its spectral radius is then $\Lambda = \max_j \Lambda_j$. Thus, if $L\,\Lambda < 1$, the synchronized solution is asymptotically stable.□

4.2 Symmetric (non commuting) Matrices

In this case we no longer have a decomposition yielding a diagonal transversal component, like (8), and therefore our conclusions are somewhat weaker than the previous theorem. Since each C_t is symmetric, doubly stochastic and irreducible, $\lambda = 1$ is the dominant eigenvalue with corresponding eigenvector $v = \{1, \ldots, 1\}$. Moreover the $(d-1)-$dimensional subspace $W = (\mathbb{R}v)^\perp$ is invariant for all C_t. This provides a decomposition like (2)

$$B_t = P^{-1}\begin{bmatrix} 0 & \\ & A_t \end{bmatrix} P \tag{10}$$

Accordingly the transversal component of (7) now satisfies

$$w_{t+1} = f'(x_t)\,(I - \mu_t\, A_t)\, w_t \tag{11}$$

The following theorem is the analogue of Theorem 1 in the present situation

Theorem 4 *Consider the system (7) where f and μ_t are as in Theorem 3 and assume $\{C_t\}$ as above. For each (μ_0, s_0), define*

$$\Lambda_n(\mu_0, s_0) = \prod_{k=0}^{n-1} \left(I - g^k(\mu_0) A_{h^k(s_0)}\right)$$

Assume that $\ln^+ ||I - \mu\, A_{h(s)}||$ belongs to $L^1(\nu \times \eta)$. Then there exists a set $F \subset [0,1] \times [0,1]$ of full $\nu \times \eta$-measure such that for all $(\mu_0, s_0) \in F$, the limit $\Lambda(\mu_0, s_0) = \lim_n ||\Lambda_n(\mu_0, s_0)||^{1/n}$ exists. Moreover if $\Lambda = \sup_{(\mu_0,s_0)} \Lambda(\mu_0, s_0)$ satisfies $L\,\Lambda < 1$, the synchronized solution of (7) is asymptotically stable.

Proof: Consider the map $\mathcal{Q}$ on $[0,1] \times [0,1]$, given by $\mathcal{Q}(\mu, s) = (g(\mu), h(s))$. Clearly $\nu \times \eta$ is $\mathcal{Q}$ invariant and by assumption, $\Lambda_n(\mu, s)$ above defines a $(\nu \times \eta)$-measurable cocycle. The existence of $\Lambda(\mu_0, s_0)$ for (μ_0, s_0) on a set of full measure then follows from Oseledec's ergodic theorem applied to $\Lambda_n(\mu, s)$. We also have that for a.e. (μ_0, s_0)

$$\Lambda(\mu_0, s_0) \leq \exp\left(\int_{[0,1]\times[0,1]} \ln^+ ||I - \mu A_{h(s)}||\; d(\nu \times \eta)(\mu, s)\right) \tag{12}$$

The proof of (12) follows the same path as the proof of (5), thus we will omit the details. Note that since each A_t is symmetric, there exist $\lambda(t) \in \sigma(A_t)$ for which $||I - \mu\, A_t|| = |1 - \mu\lambda(t)|$. $\lambda(t)$ is the lowest eigenvalue of C_t. If, in addition, the measure $\nu \times \eta$ is ergodic then $\Lambda(\mu_0, s_0)$ is independent of (μ_0, s_0) and in this case we have

$$\Lambda = \exp\left(\int_{[0,1]\times[0,1]} \ln\,|1 - \mu\,\lambda(h(s))|\; d(\nu \times \eta)(\mu, s)\right)$$

as a consequence of Birkhoff's theorem. As in the previous theorems, the condition $L\,\Lambda < 1$ implies the asymptotic stability of the synchronized solution. $\square$

4.3 Remarks:

(i) The assumptions on the continuity of g and h above can be relaxed. All that is needed is that each map possesses an invariant probability measure, ν and η.

(ii) Similarly to the density dependent migration, the results above *do not* extend to the $d = \infty$ case even though once again the effect of d is encoded in the joint spectrum of $\{C_t\}$.

4.4 Examples:

Some special cases of (9) and (12) are worth mentioning.

(i) if s_0 is a periodic point for h, that is, $h^p(s_0) = s_0$, and we take $\eta = \frac{1}{p}\sum_{k=0}^{p-1} \delta_{h^k(s_0)}$, then (9) becomes

$$\Lambda_j = \exp\left(\frac{1}{p}\sum_{k=0}^{p-1} \int_{[0,1]} \ln|1 - \mu\,\lambda_j(s_k)|\; d\nu(\mu)\right)$$

A similar expression corresponding to the case of a periodic point for g also holds.

(ii) if $g = h$, and $\eta = \nu$, but $\mu_0 \neq s_0$, we get for (9)

$$\Lambda_j = \exp\left(\int_{[0,1]\times[0,1]} \ln|1 - \mu\,\lambda_j(s)|\; d(\nu\times\nu)(\mu, s)\right)$$

(iii) if in (ii) we have, in addition, $\mu_0 = s_0$, that is, the term $\mu_t\, A_t$ in (11) is of the form $g^t(s_0)\, A_{g^t(s_0)}$, then (9) becomes

$$\Lambda_j = \exp\left(\int_{[0,1]} \ln|1 - s\,\lambda_j(s)|\; d\nu(s)\right)$$

Similar corresponding expressions can be obtained, without difficulty, to represent (12) in the special cases above. We leave the details as well as the formulation of other special instances of (12) to the interested reader. These formulas are useful in cases where ν and η are known, say Lebesgue, for they permit the exact calculations of the Lyapunov numbers of the transversal dynamical system.

5 Applications

In [3] we apply the above results to several special cases of interest in population dynamics. In this section we will restrict ourselves to the density dependent system (1) and leave the corresponding formulations of the results related to the time dependent system (7) to the interested reader.

A direct problem consists of determining stability once f, φ and C are given. In such situations we compute the eigenvalues $\lambda_1, \ldots, \lambda_d$ of C. Discarding the eigenvalue 1, we evaluate the integrals defining the spectrum of $\Lambda(x)$ above by Birkhoff's ergodic theorem

$$\int_0^\infty \ln|1-\lambda\varphi'(s)|\, d\rho(s) = \lim_n \frac{1}{n}\sum_{k=0}^{n-1} \ln|1-\lambda\varphi'(f^k(x))| \tag{13}$$

then the maximum of these values gives the spectral radius Λ and we can then determine stability once the Lyapunov exponent for f is known. We note that in order to use (13) we need the $f-$invariant measure, ρ, to be absolutely continuous with respect to Lebesgue measure. Such measures, called physical or SRB (Sinai-Ruelle-Bowen) measures, are known to exist in certain important cases, such as expansive maps, piecewise monotonic maps of the interval as well as *Axiom A* diffeomorphisms, even though a recent result of Bochi-Yoccoz shows that they are not typical in the C^1-topology. Further information can be found in [26], [17], [18] and [15].

One feature of Theorems 1 and 2 is their continuous dependence on φ', that is, C^1-uniform topology. This follows directly from the formulas for Λ above. In [3] we give examples that show that this dependence is *not* continuous in φ in the C^0-topology.

We can also make use of the above results to study an inverse problem. In this case f and φ are given and one is interested in finding a double stochastic matrix C, if possible, yielding the desired stability behavior for the synchronized solution of (1). In this regard, Theorem 6 below gives a partial result for symmetric matrices. We will make use of the following result from [12].

Proposition 5 *The following are corollaries 7 and 8 from [12].*

(i) *If* $\{\lambda_2, \ldots, \lambda_d\} \in [-1/(d-1), 1]$*, then there exists a symmetric doubly stochastic matrix* C *with* $\sigma(C) = \{1, \lambda_2, \ldots, \lambda_d\}$*.*

(ii) *If* $\lambda \in (-1, 1]$*, there exists a positive symmetric doubly stochastic matrix* C *such that* $\lambda \in \sigma(C)$*.*

The proof of Proposition 5 in [12] provides algorithms to find the matrices C in (i) and (ii) above.

Theorem 6 *Let f and φ be as in Theorems 1 and 2. Let L denote the Lyapunov exponent of the one dimensional map f. For $\lambda \in [0, 2]$, define*
$F(\lambda) = \exp\left(\int_0^\infty \ln|1 - \lambda\varphi'(s)|\, d\rho(s)\right)$, *$m = \inf F(\lambda)$ and $M = \sup F(\lambda)$. Then*

(i) *if $L\, m > 1$ the synchronized solution of (1) is unstable for* all *symmetric configurations C.*

(ii) *if $L\, M < 1$, the synchronized solution of (1) is stable for* all *symmetric configurations C.*

(iii) *if $L \in (\frac{1}{M}, \frac{1}{m})$, it is possible to find a symmetric doubly stochastic matrix C such that the synchronized solution of (1) has a prescribed stability behavior.*

Proof: Parts (i) and (ii) are direct consequences of Theorem 2 and the fact that $F(\lambda)$ gives the spectrum of $F(A)$. To prove (iii) we first note that any non negative symmetric doubly stochastic matrix has its spectrum contained in $(-1, 1]$. This follows from Gershgorin's theorem [7] which states that $\sigma(C)$ is contained in the set $\{\lambda \in \mathbb{C} : \forall i, |\lambda - c_{ii}| \leq \sum_{j \neq i} |c_{ij}|\}$. The hypothesis on C then imply that $\sigma(C) \subset [-1, 1]$. If C is positive, it is not difficult to show that -1 cannot be an eigenvalue of C. Since $B = I - C$, we have $\sigma(B) \subset [0, 2)$. Theorem 6 now follows easily from Theorem 2 and Proposition 5.$\square$

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